On Relations of Hyperelliptic Weierstrass al Functions

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Abstract

We study relations of the Weierstrass's hyperelliptic al-functions over a non-degenerated hyperelliptic curve $y^2 = f(x)$ of arbitrary genus g as solutions of sine-Gordon equation using Weierstrass's local parameters, which are characterized by two ramified points. Though the hyperelliptic solutions of the sine-Gordon equation had already obtained, our derivations of them is simple; they need only residual computations over the curve and primitive matrix computations.

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§1. Introduction

The sine-Gordon equation is a famous nonlinear integrable differential equations. For a hyperelliptic curve C_g ($y^2 = f(x) = (x - b_1)\cdots(x - b_{2g+1})$) of genus g, the hyperelliptic solutions of the sine-Gordon equation were formulated in [Mu 3.241] in terms of Riemann theta functions. In [Mu], (U, V, W) representation of symmetric product space of the g curves $Symm^g C_g$ is defined; especially, U is defined by $U(z) := (x_1 - z)\cdots(x_g - z)$ a for a point $((x_1, y_1), \cdots, (x_g, y_g))$ in $Symm^g C_g$. (In this article, we will denote U by $F(z)$ on later following the conventions in [Ba1, 2, 3, Ma].) Using the relation between U and the Riemann theta functions in [Mu 3.113], the solutions [Mu 3.241] can be rewritten as,

$$\frac{\partial}{\partial t_{\mathrm{P}}}\frac{\partial}{\partial t_{\mathrm{Q}}}\log([2\mathrm{P} - 2\mathrm{Q}]) = A([2\mathrm{P} - 2\mathrm{Q}] - [2\mathrm{Q} - 2\mathrm{P}]), \qquad (1\text{-}3)$$

where P and Q are ramified points of C_g, A is a constant number, $[D]$ is a meromorphic function over $\mathrm{Sym}^g(C_g)$ with a divisor D for each C_g and $t_{\mathrm{P}'}$ is a coordinate in the Jacobi

variety such that it is identified with a local parameter at a branch point P' up to constant. In other words, for a finite branch point $(b_i, 0)$ $U(b_i)$ is identified with $[2(b_i,0) - 2\infty]$ up to constant factor.

In the formulations in [Mu], local parameters $t_{\mathrm{P}'}$ were not concretely treated. In this article, we will give more explicit representations of (1-3) using concrete local parameters in [Ba2, W2, 3] and present simpler derivations of (1-3) without using any θ-function. This article is an application of a scheme developed in [Ma] to the sine-Gordon equation, which is based upon [Ba3].

In [W1, W2], Weierstrass defined al function by $\mathrm{al}_r = \gamma_r \sqrt{U(b_r)}$ using a constant factor γ_r. In Theorem 3.1, we will give

$$\frac{\partial^2}{\partial u_1^{(r)} \partial u_g^{(r)}} \log \mathrm{al}_r(u^{(r)}) = \frac{1}{2}\left(\frac{\mathrm{al}_r^2(u^{(r)})}{\gamma_r^2} - \frac{f'(b_r)\gamma_r^2}{\mathrm{al}_r^2(u^{(r)})} \right), \tag{1-4}$$

in terms of a coordinate system $u^{(r)}$'s defined in (2-5). $\mathrm{al}_r(u)$ has the single order zero at $(b_r, 0)$ and a singularity of the single order at ∞ as a function of $x_i \in C_g$.

Further we give another representation in Theorem 4.1 in terms of v's defined in (2-6) $(a_1 := b_r, a_2 := b_s)$ [W2, 3],

$$\frac{\partial^2}{\partial v_1 \partial v_2} \log \frac{\mathrm{al}_r(v)}{\mathrm{al}_s(v)} = \frac{1}{2(b_r - b_s)} \left(f'(b_s) \frac{\gamma_s^2 \mathrm{al}_r(v)^2}{\gamma_r^2 \mathrm{al}_s(v)^2} + f'(b_r) \frac{\gamma_r^2 \mathrm{al}_s(v)^2}{\gamma_s^2 \mathrm{al}_r(v)^2} \right). \tag{1-5}$$

The function $\mathrm{al}_s(v)/\mathrm{al}_r(v)$ vanishes with order one when x_i is at $(b_s, 0)$ whereas it diverges with order one if x_i approaches to $(b_r, 0)$. As they were discovered by Weierstrass [W2, 3] and they play the essential roles in the investigation in [W2, 3] and in §4. Thus we have called them *Weierstrass parameters.*

In these proofs, we will use only residual computations using the data of curve itself without any θ functions as the derivation of hypereilliptic solutions of the modified Korteweg-de Vries equations in [Ma]. The curve is sometimes given by an affine equation with special coefficients. Then it might be important to study the relation between the properties of line-bundle over the curve and these coefficients. As (1-4) and (1-5) can be explicitly expressed by data of curve C_g, the author believes that they have some advantage as relations of special functions.

§2. Differentials of a Hyperelliptic Curve

In this section, we will give the conventions and notations of the hyperelliptic functions in this article. We denote the set of complex numbers by $\mathbb{C}$ and the set of integers by $\mathbb{Z}$.

2.1 Hyperelliptic Curve. We deal with a hyperelliptic curve C_g of genus g ($g > 0$) given by the affine equation,

$$\begin{aligned} y^2 &= f(x) \\ &= \lambda_{2g+1}x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ &= (x - b_r)h_r(x), \end{aligned} \tag{2-1}$$

where $\lambda_{2g+1} \equiv 1$ and λ_j's are complex numbers. We use the expressions,

$$\begin{aligned} f(x) &:= (x - b_1)(x - b_2)\cdots(x - b_{2g})(x - b_{2g+1}) \\ &= P(x)Q(x), \\ P(x) &:= (x - a_1)(x - a_2)\cdots(x - a_g), \\ Q(x) &:= (x - c_1)(x - c_2)\cdots(x - c_g)(x - c), \end{aligned} \tag{2-2}$$

where b_j's ($b_i = a_i, b_{g+i} = c_i$) are complex numbers.

It is noted that the permutation group acts on these $\{b_r\}$ and $\{a_r\}$.

2.2 Definition [Ba1 , Ba2, W2, 3]**.**

(1) For a point $(x_i, y_i) \in C_g$, the unnormalized differentials of the first kind are defined by,

$$du_1^{(r,i)} := \frac{dx_i}{2y_i}, \quad du_2^{(r,i)} := \frac{(x_i - b_r)dx_i}{2y_i}, \quad \cdots, \quad du_g^{(r,i)} := \frac{(x_i - b_r)^{g-1}dx_i}{2y_i}. \tag{2-3}$$

$$dv_1^{(i)} := \frac{P(x_i)dx_i}{2P'(a_1)(x_i - a_1)y_i}, \quad dv_2^{(i)} := \frac{P(x_i)dx_i}{2P'(a_2)(x_i - a_2)y_i}, \quad \cdots, \quad dv_g^{(i)} := \frac{P(x_i)dx_i}{2P'(a_g)(x_i - a_g)y_i}. \tag{2-4}$$

(2) Let us define the Abel maps for g-th symmetric product of the curve C_g,

$$u^{(r)} := (u_1^{(r)}, \cdots, u_g^{(r)}) : \mathrm{Sym}^g(C_g) \longrightarrow \mathbb{C}^g,$$

$$\left(u_k^{(r)}((x_1,y_1),\cdots,(x_g,y_g)) := \sum_{i=1}^{g} \int_{\infty}^{(x_i,y_i)} du_k^{(r,i)}\right), \tag{2-5}$$

$$v := (v_1,\cdots,v_g) : \mathrm{Sym}^g(C_g) \longrightarrow \mathbb{C}^g,$$

$$\left(v_k((x_1,y_1),\cdots,(x_g,y_g)) := \sum_{i=1}^{g} \int_{\infty}^{(x_i,y_i)} dv_k^{(i)}\right). \tag{2-6}$$

These coordinates are universal covering of the related Jacobian $\mathcal{J}$. The definition (2-6) [Ba2 p.382] is due to Weierstrass [W2, 3] and we call (2-6) *Weierstrass parameter*, though we choose different constant factor from the original one [W2, 3]. This parameterization is a key of the second solutions mentioned in §4.

2.3 Definition.

(1) Hyperelliptic *al* function is defined by [Ba2 p.340, W2, 3],

$$\mathrm{al}_r(u) := \gamma_r \sqrt{F(b_r)}, \tag{2-7}$$

where $\gamma_r := \sqrt{-1/P'(b_r)}$ and

$$\begin{aligned} F(x) &:= (x-x_1)\cdots(x-x_g) \\ &= (x-b_r-x_1+b_r)\cdots(x-b_r-x_g+b_r). \end{aligned} \tag{2-8}$$

On the choice of γ_r, we will employ the convention of Baker [Ba2] instead of original one [W2, 3]. We note that al_r's have mutually algebraic relations.

For later convenience, a polynomial associated with $F(x)$ is introduced by

$$\begin{aligned} \pi_i^{(r)}(x) &:= \frac{F(x)}{x-x_i} \\ &= \chi_{i,g-1}^{(r)}(x-b_r)^{g-1} + \chi_{i,g-2}^{(r)}(x-b_r)^{g-2} + \cdots + \chi_{i,1}^{(r)}(x-b_r) + \chi_{i,0}^{(r)}. \end{aligned}$$

Then we have $\chi_{i,g-1}^{(r)} \equiv 1$ and $\chi_{i,0}^{(r)} = F(b_r)/(x_i - b_r)$. Further we introduce $g\times g$-matrices,

$$\mathcal{W}^{(r)} := \begin{pmatrix} \chi_{1,0}^{(r)} & \chi_{1,1}^{(r)} & \cdots & \chi_{1,g-1}^{(r)} \\ \chi_{2,0}^{(r)} & \chi_{2,1}^{(r)} & \cdots & \chi_{2,g-1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{g,0}^{(r)} & \chi_{g,1}^{(r)} & \cdots & \chi_{g,g-1}^{(r)} \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_g \end{pmatrix},$$

$$\mathcal{M} := \begin{pmatrix} \frac{1}{x_1 - a_1} & \frac{1}{x_2 - a_1} & \cdots & \frac{1}{x_g - a_1} \\ \frac{1}{x_1 - a_2} & \frac{1}{x_2 - a_2} & \cdots & \frac{1}{x_g - a_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_1 - a_g} & \frac{1}{x_2 - a_g} & \cdots & \frac{1}{x_g - a_g} \end{pmatrix},$$

$$\mathcal{P} = \begin{pmatrix} \sqrt{\frac{P(x_1)}{Q(x_1)}} & & & \\ & \sqrt{\frac{P(x_2)}{Q(x_2)}} & & \\ & & \ddots & \\ & & & \sqrt{\frac{P(x_g)}{Q(x_g)}} \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} P'(a_1) & & & \\ & P'(a_2) & & \\ & & \ddots & \\ & & & P'(a_g) \end{pmatrix}, \quad \mathcal{F}' = \begin{pmatrix} F'(x_1) & & & \\ & F'(x_2) & & \\ & & \ddots & \\ & & & F'(x_g) \end{pmatrix},$$

where $F'(x) := dF(x)/dx$.

2.3 Lemma. *For these matrices, following relations hold:*

(1) *The inverse matrix of* $\mathcal{W}^{(r)}$ *is given by* $\mathcal{W}^{(r)-1} = \mathcal{F}^{(r)\prime -1}\mathcal{V}^{(r)}$, *where* $\mathcal{V}^{(r)}$ *is Vandermond matrix,*

$$\mathcal{V}^{(r)} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ (x_1 - b_r) & (x_2 - b_r) & \cdots & (x_g - b_r) \\ (x_1 - b_r)^2 & (x_2 - b_r)^2 & \cdots & (x_g - b_r)^2 \\ \vdots & \vdots & & \vdots \\ (x_1 - b_r)^{g-1} & (x_2 - b_r)^{g-1} & \cdots & (x_g - b_r)^{g-1} \end{pmatrix}.$$

(2)

$$\det \mathcal{M} = \frac{(-1)^{g(g-1)/2}\mathrm{P}(x_1, \cdots, x_g)\mathrm{P}(a_1, \cdots, a_g)}{\prod_{k,l}(x_k - a_l)},$$

where

$$\mathrm{P}(z_1, \cdots, z_g) := \prod_{i<j}(z_i - z_j).$$

(3)

$$(\mathcal{M}\mathcal{P})^{-1}\mathcal{A} = \left[\left(\frac{2y_i F(a_j)}{F'(x_i)(a_j - x_i)}\right)_{i,j}\right]. \tag{2-9}$$

Proof. (1) is obtained by direct computations. (2) is a well-known result [T]. Since the zero and singularity in the left hand side give the right hand side as

$$C\mathrm{P}(x_1,\cdots,x_g)\mathrm{P}(a_1,\cdots,a_g)/\prod_{k,l}(x_k-a_l),$$

for a certain constant C. In order to determine C, we multiply $\prod_{k,l}(x_k-a_l)$ both sides and let $x_1=a_1$, $x_2=a_2$, $\cdots$, and $x_g=a_g$. Then C is determined as above. (3) is obtained by the Laplace formula using the minor determinant for the inverse matrix.∎

Then we have following corollary.

2.5 Corollary. *Let* $\partial^{(r)}_{u_i}:=\partial/\partial u^{(r)}_i$, $\partial_{v_i}:=\partial/\partial v_i$, *and* $\partial_{x_i}:=\partial/\partial x_i$.

$$\begin{pmatrix}\partial^{(r)}_{u_1}\\ \partial^{(r)}_{u_2}\\ \vdots\\ \partial^{(r)}_{u_g}\end{pmatrix}=2\mathcal{Y}\mathcal{F}'^{-1}\cdot{}^t\mathcal{W}^{(r)}\begin{pmatrix}\partial_{x_1}\\ \partial_{x_2}\\ \vdots\\ \partial_{x_g}\end{pmatrix},\quad \begin{pmatrix}\partial_{v_1}\\ \partial_{v_2}\\ \vdots\\ \partial_{v_g}\end{pmatrix}=2(\mathcal{M}\mathcal{P})^{-1}\mathcal{A}\begin{pmatrix}\partial_{x_1}\\ \partial_{x_2}\\ \vdots\\ \partial_{x_g}\end{pmatrix}. \tag{2-10}$$

§3. Relations between Hyperelliptic al Functions (b_r,∞)-type

In this section, we will give the first relation of hyperelliptic al function using the parameters $u^{(r)}_1$ and $u^{(r)}_g$ in (2-5).

3.1 Theorem.

$$\frac{\partial}{\partial u^{(r)}_1}\frac{\partial}{\partial u^{(r)}_g}\log \mathrm{al}_r=\frac{1}{2}\left(\frac{\mathrm{al}_r^2}{\gamma_r^2}-\frac{f'(b_r)\gamma_r^2}{\mathrm{al}_r^2}\right). \tag{3-1}$$

Here we will give a comment on Theorem 3.1. Let us fix the parameters $x_2,\cdots,x_g$ and regard al_r as a function of a parameter related to x_1 over C_g. Then its divisor is $(\mathrm{al}_r)=(b_r,0)-\infty$. Further by letting $t^2=(x_i-b_r)$ around $(b_r,0)$, the definition (2-3) shows,

$$du^{(r,i)}_1|_{(b_r,0)}=\frac{2}{\sqrt{f'(b_r)}}dt,$$

while for $s^2 = 1/x$ around ∞,

$$du_g^{(r,i)}|_{(\infty)} = -2ds.$$

Hence (3-1) can be regarded as an explicit representation of (1-3).

Proof. Instead of (3-1), we will prove following formula (3-2) in remainder in this section.

$$\frac{\partial}{\partial u_1^{(r)}} \frac{\partial}{\partial u_g^{(r)}} \log F(b_r) = F(b_r) - \frac{f'(b_r)}{F(b_r)}. \tag{3-2}$$

The strategy is essentially the same as [Ba3, Ma]. First we translate the words of the Jacobian into those of the curves; we rewrite the differentials $u^{(r)}$'s in terms of the differentials over curves as in (3-3). We count the residue of an integration and use a combinatorial trick. Then we will obtain (3-2).

From (2-10), we will express $u^{(r)}$'s by the affine coordinates x_i's,

$$\frac{\partial}{\partial u_g^{(r)}} = \sum_{i=1}^{g} \frac{2y_i}{F'(x_i)} \frac{\partial}{\partial x_i},$$

$$\frac{\partial}{\partial u_1^{(r)}} = \sum_{i=1}^{g} \frac{2y_i \chi_{i,0}^{(r)}}{F'(x_i)} \frac{\partial}{\partial x_i} = F(b_r) \sum_{i=1}^{g} \frac{2y_i}{(x_i - b_r)F'(x_i)} \frac{\partial}{\partial x_i}. \tag{3-3}$$

Hence the right hand side of (3-2) becomes

$$-\frac{\partial^2}{\partial u_1^{(r)} \partial u_g^{(r)}} \log F(b_r) = F(b_r) \sum_{j=1,i=1}^{g} \frac{2y_j}{(x_i - b_r)^2 F'(x_j)} \frac{\partial}{\partial x_j} \frac{2y_i}{F'(x_i)(x_i - b_r)^2}. \tag{3-4}$$

Here we will note the derivative of $F(x)$, which is shown by direct computations.

$$\frac{\partial}{\partial x_k} \left(\left[\frac{\partial}{\partial x} F(x) \right]_{x=x_k} \right) = \frac{1}{2} \left[\frac{\partial^2}{\partial x^2} F(x) \right]_{x=x_k}.$$

Then (3-4) can be written as,

$$-\frac{\partial}{\partial u_1^{(r)}} \frac{\partial}{\partial u_g^{(r)}} \log F(b_r) = F(b_r) \sum_{i=1}^{g} \frac{1}{F'(x_i)} \left[\frac{\partial}{\partial x} \left(\frac{f(x)}{(x - b_r)F'(x)} \right) \right]_{x=x_i}$$
$$-F(b_r) \sum_{k,l,k\neq l} \frac{4y_k y_l}{(b_r - x_k)(b_r - x_l)(x_k - x_l)F'(x_k)F'(x_l)}.$$

The proof of Theorem 3.1 finishes due to the following lemma. ∎

3.2 Lemma. *Following relations hold:*

$$\sum_{k=1}^{g} \frac{1}{F'(x_k)} \left[\frac{\partial}{\partial x}\left(\frac{f(x)}{(x-b_r)^2 F'(x)}\right)\right]_{x=x_k} = 1 - \frac{f'(b_r)}{F(b_r)^2}. \tag{3-5}$$

$$\sum_{k,l,k\neq l} \frac{2y_k y_l}{(b_r-x_k)(b_r-x_l)(x_k-x_l)F'(x_k)F'(x_l)} = 0. \tag{3-6}$$

Proof. : (3-5) will be proved by the following residual computations: Let ∂C_g^o be the boundary of a polygon representation C_g^o of C_g,

$$\oint_{\partial C_g^o} \frac{f(x)}{(x-b_r)^2 F(x)^2} dx = 0. \tag{3-7}$$

The divisor of the integrand of (3-7) is given by,

$$\left(\frac{f(x)}{(x-b_r)^2 F(x)^2} dx\right) = 3 \sum_{i=1, b_i\neq b_r}^{2g+1} (b_i, 0) - (b_r, 0) - 2\sum_{i=1}^{g}(x_i, y_i) - 2\sum_{i=1}^{g}(x_i, -y_i) - \infty.$$

We check these poles: First we consider the contribution around ∞ point. Noting that the local parameter t at ∞ is $x = 1/t^2$,

$$\mathrm{res}_\infty \frac{f(x)}{(x-b_r)^2 F(x)^2} dx = -2.$$

Since the local parameter t at $(x_k, \pm y_k)$ is $t = x - x_k$, we have

$$\mathrm{res}_{(x_k, \pm y_k)} \frac{f(x)}{(x-b_r)^2 F(x)^2} dx = \frac{1}{F'(x_k)} \left[\frac{\partial}{\partial x}\left(\frac{f(x)}{(x-b_r)^2 F'(x)}\right)\right]_{x=x_k}.$$

For each branch point $(b_r, 0)$, the local parameter t is $t^2 = x - b_r$ and thus

$$\mathrm{res}_{(b_r,0)} \frac{f(x)}{(x-b_r)^2 F(x)^2} dx = 2\frac{f'(b_r)}{F(b_r)^2}.$$

By arranging them, we obtain (3-5).

On the other hand, (3-6) can be proved by using a trick: for $i \neq j$,

$$\frac{1}{(b_r-x_k)(b_r-x_l)(x_k-x_l)} = \frac{1}{(x_k-x_l)^2}\left(\frac{1}{(b_r-x_k)} - \frac{1}{(b_r-x_l)}\right).$$

∎

§4. Relations between Hyperelliptic al Functions: (a_1, a_2)-type

In the previous section, we have a solution with a duality between a finite ramified point and ∞-point. In this section, we will give a relation between hyperelliptic al functions using the Weierstrass parameter (2-6). The relation has a duality between finite ramified points $(a_r, 0)$ and $(a_s, 0)$.

4.1 Theorem. *For $r \neq s$, we obtain*

$$\frac{\partial}{\partial v_r}\frac{\partial}{\partial v_s}\log\frac{\mathrm{al}_r}{\mathrm{al}_s} = \frac{1}{2(a_r - a_s)}\left(f'(a_r)\frac{\gamma_r^2 \mathrm{al}_s^2}{\gamma_s^2 \mathrm{al}_r^2} + f'(a_s)\frac{\gamma_s^2 \mathrm{al}_r^2}{\gamma_r^2 \mathrm{al}_s^2}\right). \tag{4-1}$$

Before we prove it, we will give some comments: Let us fix the parameter $x_2, \cdots, x_g$ and regard $\mathrm{al}_r/\mathrm{al}_s(\propto \sqrt{F(a_r)/F(a_s)})$ as a function of x_1 over C_g. Then its divisor is $(\mathrm{al}_r/\mathrm{al}_s) = (a_r, 0) - (a_s, 0)$. By letting $t_r^2 = (x_i - a_r)$ around $(a_r, 0)$, infinitesimal value of Weierstrass parameter (2-4) is given,

$$dv_r^{(i)}|_{(a_r,0)} = \frac{1}{\sqrt{f'(a_r)}}dt_r.$$

Thus (4-1) is also a concrete expression of (1-3).

Proof. Similar to the proof of Theorem 3.1, let us prove the theorem. Without loss of generality, we will prove the following relation instead of (4-1):

$$\frac{\partial}{\partial v_1}\frac{\partial}{\partial v_2}\log\frac{F(a_1)}{F(a_2)} = \frac{F(a_1)F(a_2)}{(a_1 - a_2)}\left(\frac{f'(a_1)}{F(a_1)^2} + \frac{f'(a_2)}{F(a_2)^2}\right). \tag{4-2}$$

From (2-9) and (2-10), the derivative v's are expressed by the affine coordinate x_i's,

$$\frac{\partial}{\partial v_r} = F(a_r)\sum_{j=1}^{g}\frac{2y_j}{F'(x_j)(x_j - a_r)}\frac{\partial}{\partial x_j}.$$

The right hand side of (4-2) becomes,

$$\frac{\partial^2}{\partial v_1 \partial v_2}\log\frac{F(a_1)}{F(a_2)} = F(a_1)\sum_{j=1 i=1}^{g}\frac{2y_j}{(x_i - a_1)F'(x_j)}\frac{\partial}{\partial x_j}\frac{2y_i F(a_2)}{F'(x_i)(x_i - a_2)}\frac{(a_1 - a_2)}{(x_i - a_1)(x_i - a_2)}. \tag{4-3}$$

The right hand side of (4-3) is

$$F(a_1)F(a_2)\sum_{i=1}^{g}\frac{1}{F'(x_i)}\left[\frac{\partial}{\partial x}\left(\frac{f(x)(a_2-a_1)}{(x-a_1)^2(x-a_2)^2F'(x)}\right)\right]_{x=x_i}$$

$$-F(a_1)F(a_2)\sum_{k,l,k\neq l}\frac{2y_ky_l(a_2-a_1)}{F'(x_k)F'(x_l)(x_l-a_1)(x_k-a_2)(x_k-a_1)(x_l-a_2)(x_l-x_k)}.$$

Then the proof of Theorem 4.1 is completely done due to the following lemma. ∎

4.2 Lemma. *Following relations hold:*

$$\sum_{i=1}^{g}\frac{1}{F'(x_i)}\left[\frac{\partial}{\partial x}\left(\frac{f(x)}{(x-a_1)^2(x-a_2)^2F'(x)}\right)\right]_{x=x_i}=\frac{1}{(a_1-a_2)^2}\left(\frac{f'(a_1)}{F(a_1)^2}-\frac{f'(a_1)}{F(a_1)^2}\right). \tag{4-4}$$

$$\sum_{k,l,k\neq l}\frac{2y_ky_l(a_2-a_1)}{F'(x_k)F'(x_l)(x_l-a_1)(x_k-a_2)(x_k-a_1)(x_l-a_2)(x_l-x_k)}=0. \tag{4-5}$$

Proof. : Similar to Lemma 3-2, we consider an integral,

$$\oint_{\partial C_g^o}\frac{f(x)}{(x-a_1)^2(x-a_2)^2F(x)^2}dx=0. \tag{4-6}$$

As the divisor of the integrand of (4-6) is

$$\left(\frac{f(x)}{(x-a_1)^2(x-a_2)^2F(x)^2}dx\right)$$
$$=3\sum_{i=1,b_i\neq a_1,a_2}^{2g+1}(b_i,0)-(a_1,0)-(a_2,0)-2\sum_{i=1}^{g}(x_i,y_i)-2\sum_{i=1}^{g}(x_i,-y_i)+3\infty, \tag{4-9}$$

we count residual contributions from each terms as in the proof of Lemma 3-2 and obtain (4-4). Considering the symmetry, (4-5) is easily obtained. ∎

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